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LETTER TO THE EDITOR

A (2 + 1)-dimensional extension for the sine-Gordon equation

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Abstract. Starting from the breaking soliton equation, we obtain a new integrable equation in 2+1 dimensions. Though the equation has no exchange symmetry of the space variables x and y, the model reduces back to the known $(1+1)$-dimensional sine-Gordon (or Liouville) equation.

The breaking soliton equation

$$u_{xt}=4u_xu_{xy}+2u_yu_{xx}-u_{xxxy} \tag{1}$$

was first established by Calogero and Degasperis [1]. Equation (1) is used to describe the $(2+1)$-dimensional interaction of Riemann wave propagation along the y-axis with long wave propagation along the x-axis [2, 3]. Set $v=u_x$, then (1) can be written as

$$v_t=4vv_y+2(\partial_x^{-1}v_y)v_x-v_{xxy}\equiv K(v) \tag{2}$$

which reduces to the known KdV equation when we take $x=y$. The bi-Hamiltonian structure and the Lax pair equations with non-isospectral problem have been discussed in [3].

We know that starting from any one symmetry of an integrable model one can obtain a new integrable equation. For instance the well known sine-Gordon equation

$$\phi_{xt}=\sin 2\phi \tag{3}$$

can be obtained from the non-local symmetry of the KdV (or mKdV) equation [4]. A symmetry σ of the evolution equation (2) is defined as

$$\sigma_t=K'\sigma\equiv 4v_y\sigma+4v\sigma_y+2v_x\partial_x^{-1}\sigma_y+2\sigma_x\partial_y^{-1}v_y-\sigma_{xxy} \tag{4}$$

such that (2) is form invariant under the transformation

$$v\rightarrow v+\varepsilon\sigma \qquad (\varepsilon \text{ infinitesimal}). \tag{5}$$

By means of the recursion operator (2) [3] various solutions of (4) can be obtained. Here we only give a special non-local symmetry of (2) to obtain the extended

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sine-Gordon equation in 2+1 dimensions. The direct calculations tell us that

$$\sigma_0 = 2\psi_x\psi(1+\partial_x^{-1}\psi^{-3}\psi_y)+\psi^{-1}\psi_y \tag{6}$$

with

$$-\psi_{xx}+v\psi=0 \tag{7}$$

$$\psi_t=-v_y\psi+2\psi_x\partial_x^{-1}v_y \tag{8}$$

is a non-local symmetry of the breaking soliton equation (1). It is straightforward to verify that the compatibility condition $\psi_{xxt}=\psi_{txx}$ of (7) and (8) is just equation (1). Now using the symmetry σ_0 we obtain a new integrable 2|1 dimensional equation:

$$v_t=\sigma_0=2\psi_x\psi(1+\partial_x^{-1}\psi^{-3}\psi_y)+\psi^{-1}\psi_y \tag{9}$$

$$\psi_{xx}=v\psi \qquad \text{or} \qquad v=\psi_{xx}/\psi. \tag{10}$$

This equation is a (2+1)-dimensional extension of the negative (1+1)-dimensional KdV equation

$$v_t=2\psi_x\psi \qquad \psi_{xx}=v\psi. \tag{11}$$

Equation (11) is related to the sine-Gordon equation (3) and/or the Liouville equation by the well known Miura transformation [4, 5]. When $\psi_y=0$ ($v_y=0$) or $x=y$, equation (9) is reduced to (11). In order to obtain the extended 2|1 dimensional sine-Gordon equation from (9) and (10), we can substitute (10) into (9)

$$\frac{\psi_{xxt}}{\psi}-\frac{\psi_{xx}\psi_t}{\psi^2}=2\psi_x\psi(1+\partial_x^{-1}\psi^{-3}\psi_y)+\psi^{-1}\psi_y. \tag{12}$$

Now multiplying (12) by ψ^2 and integrating once with respect to x we have

$$\psi_{xt}\psi-\psi_x\psi_t=\tfrac{1}{2}\psi^4\partial_x^{-1}\psi^{-3}\psi_y+\tfrac{1}{2}\psi^4+\partial_x^{-1}\psi\psi_y+C \tag{13}$$

i.e.,

$$\left(\frac{\psi_x}{\psi}\right)_t=\tfrac{1}{2}(\psi^2\partial_x^{-1}\psi^{-3}\psi_y)+\tfrac{1}{2}(\psi^{-2}\partial_x^{-1}\psi\psi_y)+\tfrac{1}{2}\psi^2+C\psi^{-2} \tag{14}$$

where C is an integral constant. Finally, setting

$$\psi=\exp \mathrm{i}\phi \tag{15}$$

we obtain

$$\phi_{xt}=\tfrac{1}{2}\cos 2\phi\partial_x^{-1}(\sin 2\phi)_y-\tfrac{1}{2}\sin 2\phi\partial_x^{-1}(\cos 2\phi)_y+\sin 2\phi \tag{16}$$

with $C=-\frac{1}{2}$. It is clear that equation (16) is a 2×1 dimensional extension of the sine-Gordon equation (3). When $x=y$ or $\phi_y=0$, equation (15) reduces to the usual sine-Gordon equation.

Taking the integral constant C as zero and setting

$$\psi=\exp\phi \tag{17}$$

for (14), the extended (2+1)-dimensional integrable Liouville equations are obtained:

$$\phi_{xt}=\tfrac{1}{2}\exp 2\phi+\exp(-2\phi)\partial_x^{-1}\partial_y\exp 2\phi-\exp 2\phi\partial_x^{-1}\partial_y\exp(-2\phi). \tag{18}$$

It is known that there exist different (2+1)-dimensional extensions for the KdV equation. The Kadomtsev–Petviashvili equation [6] and the Veselov–Novikov equation [7] are two significant examples. In the same way, there may be different (2+1)-dimensional extensions for the sine-Gordon equations. The study of other types of (2+1)-dimensional sine-Gordon extension are in progress. More information about the extended sine-Gordon equation (16), such as the soliton solutions, infinite many symmetries, conservation laws and other integrable properties, are also worth further study.

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